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A Pointed Weak Energy Conservation Law via Noether's Theorem

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Abstract: Noether's theorem is used to identify a conservation law for the quantum mechanical pointed weak energy. Under this law the pointed weak energy is a constant of the motion and leads to simple expressions for the correlation amplitude between and probability for an associated evolving quantum state and the state at a previous initial time.

Keywords: Noether's Theorem; pointed weak energy; conservation law.

Introduction

A complex valued energy has recently been defined for pre-and post-selected (PPS) quantum systems and studied from a theoretical perspective. Because the mathematical form of this quantity is the same as that of a weak value (Aharonov and Vaidman 1990; Aharonov *et al* 1986 ; Aharonov *et al* 1988) it is called the weak energy. It has been shown that the weak energy of an evolving PPS system can be expressed as a simple homogeneous Lagrangian energy function in terms of a Pancharatnam (P) phase χ and a Fubini-Study (FS) distance s associated with the PPS states. Additional properties associated with weak energy can be found in (Parks 2014).

The notion of pointed weak energy was introduced in (Parks 2006). This is the weak energy related to the evolution of a quantum state relative to its fixed initial state. In (Parks 2006) several of its properties, e.g. a U(1) gauge potential, integral invariants, etc., were identified and discussed and in (Parks 2007) its relationship to quantum geometric phase was established. Similar to weak energy, pointed weak energy is also expressed as a simple homogeneous Lagrangian in terms of χ and s. It is denoted $\mathcal{L}$, called the PFS Lagrangian (to distinguish it from the weak energy Lagrangian), and defines the PFS functional

$$I = \int \mathcal{L}\, dt\,.$$

In this paper it is shown that I is both an extremal and divergence invariant under infinitesimal transformations of time t, χ, and s. This enables application of Noether's theorem (see Appendix) to provide a conservation law, the physical consequences of which are briefly discussed.

Preliminaries

Let $|\psi_0\rangle$ be a distinguished state in a Hilbert space $\mathcal{H}$ with projective space $\wp$ comprised of all the rays of $\mathcal{H}$ (a ray is an equivalence class $[\psi]$ of states of $|\psi\rangle$ in $\mathcal{H}$ which differ only in phase) and with projection map $\Pi: \mathcal{H} \to \wp$ such that $|\psi\rangle \mapsto [\psi]$. Furthermore, let $\mathcal{H}_{\not\perp} = \{|\psi\rangle \in \mathcal{H}: \langle\psi|\psi_0\rangle \neq 0\}$ and define the pointed map $\Psi_0: \mathcal{H}_{\not\perp} \to R \times R$ by

$$\Psi_0(|\psi\rangle) = \left(arg\frac{\langle\psi|\psi_0\rangle}{|\langle\psi|\psi_0\rangle|}, 2\sqrt{1-|\langle\psi|\psi_0\rangle|^2} \right)$$
$$= (\chi, s),$$

where R is the set of real numbers and $\chi \in [0,2\pi)$ is the Pancharatnam phase defined by

$$e^{i\chi} = \frac{\langle\psi|\psi_0\rangle}{|\langle\psi|\psi_0\rangle|}\,. \tag{1}$$

The quantity

$$s = 2\sqrt{1 - |\langle\psi|\psi_0\rangle|^2} \in [0,2) \qquad (2)$$

is the generalized Fubini-Study metric distance separating $[\psi]$ and $[\psi_0]$ in $\wp$. The PFS configuration space associated with any distinguished state $|\psi_0\rangle$ is the image set $im\,\Psi_0 = [0,2\pi) \times [0,2) \subset R \times R$ with origin $\Psi_0(|\psi_0\rangle) = (0,0)$. The quantity $|\langle\psi|\psi_0\rangle|^2$ is the correlation probability between states $|\psi\rangle$ and $|\psi_0\rangle$. It is assumed that the evolution of a time dependent state $|\psi(t)\rangle$ occurs continuously, smoothly, and entirely within $\mathcal{H}_\psi$, with $|\psi_0\rangle = |\psi(0)\rangle$.

The pointed weak energy $W_0(t)$ for the normalized state $|\psi(t)\rangle$ at time t is defined by

$$W_0(t) = \frac{\langle\psi(t)|\hat{H}|\psi(0)\rangle}{\langle\psi(t)|\psi(0)\rangle}$$

$$= Re\,W_0(t) + i\,Im\,W_0(t), \qquad (3)$$

where

$$i\hbar\frac{d|\psi(t)\rangle}{dt} = \hat{H}|\psi(t)\rangle\,.$$

It is readily determined from (1) and (2) that

$$Re\,W_0(t) = \hbar\frac{d\chi(t)}{dt}$$

and

$$Im\,W_0(t) = \hbar\left[\frac{s(t)}{4 - s^2(t)}\right]\left(\frac{ds(t)}{dt}\right).$$

Upon substitution of these equalities into (3), the pointed weak energy assumes the form of the PFS Lagrangian

$$\mathcal{L} \equiv \hbar\frac{d\chi(t)}{dt} + i\hbar\left[\frac{s(t)}{4 - s^2(t)}\right]\left(\frac{ds(t)}{dt}\right) \qquad (4)$$

with

$$\hbar = \frac{\partial\mathcal{L}}{\partial(d\chi(t)/dt)} \equiv p_\chi \qquad (5)$$

as the generalized momentum conjugate to χ and

$$i\hbar\left[\frac{s(t)}{4 - s^2(t)}\right] = \frac{\partial\mathcal{L}}{\partial(ds(t)/dt)} \equiv p_s \qquad (6)$$

as the generalized momentum conjugate to s.

The Divergence Invariance of *I*

The divergence invariance of I requires that, under the infinitesimal transformations

$$\begin{cases} t' = t + \epsilon\gamma \\ \chi'(t) = \chi(t) + \epsilon\alpha\,, \\ s'(t) = s(t) + \epsilon\beta \end{cases} \qquad (7)$$

with generators α, β, and γ and infinitesimally small number ϵ, there must exist a function $f \equiv f(t)$ such that

$$\mathcal{L}'\left(\frac{dt'}{dt}\right) - \mathcal{L} = \epsilon\frac{df}{dt} + O(\epsilon^r),\ \ r > 1, \qquad (8)$$

where

$$\mathcal{L}' \equiv \hbar\frac{d\chi'(t')}{dt'} + i\hbar\frac{s'(t')}{4 - s'(t')^2}\frac{ds'(t')}{dt'}. \qquad (9)$$

To determine if I is divergence invariant first substitute the transformations in (7) into $\chi'(t')$ and $s'(t')$ to find:

$$\chi'(t') = \chi'(t + \epsilon\gamma) = \chi'(t) + \left.\frac{d\chi'(t')}{dt'}\right|_0 \epsilon\gamma + \mathcal{O}(\epsilon^2)$$

or

$$\chi'(t') = \chi(t) + \epsilon(\alpha + \eta\gamma) + \mathcal{O}(\epsilon^2),$$

$$s'(t') = s'(t + \epsilon\gamma) = s'(t) + \left.\frac{ds'(t')}{dt'}\right|_0 \epsilon\gamma + \mathcal{O}(\epsilon^2)$$

or

$$s'(t') = s(t) + \epsilon(\beta + \sigma\gamma) + \mathcal{O}(\epsilon^2),$$

$$\frac{d\chi'(t')}{dt'} = \frac{d\chi'(t')}{dt}\frac{dt}{dt'}$$

$$= \left(\frac{d\chi(t)}{dt} + \epsilon\left\{\frac{d\alpha}{dt} + \frac{d(\eta\gamma)}{dt}\right\}\right)\frac{dt}{dt'} + O(\epsilon^2)\,, \qquad (10)$$

and similarly

$$\frac{ds'(t')}{dt'} = \left(\frac{ds(t)}{dt} + \epsilon\left\{\frac{d\beta}{dt} + \frac{d(\sigma\gamma)}{dt}\right\}\right)\frac{dt}{dt'}$$

$$+O(\epsilon^2). \qquad (11)$$

Here the notation $X|_0$ means X evaluated at $\epsilon = 0$, i.e. at $t' = t$. Also,

$$\frac{s'(t')}{4 - s'(t')^2} = \frac{s(t) + \epsilon\{\beta + \sigma\gamma\} + O(\epsilon^2)}{4 - \left(s(t) + \epsilon\{\beta + \sigma\gamma\} + O(\epsilon^2)\right)^2}$$

$$= \frac{s(t) + \epsilon\{\beta + \sigma\gamma\} + O(\epsilon^2)}{4 - \left(s(t) + \epsilon\{\beta + \sigma\gamma\} + O(\epsilon^2)\right)^2}$$

$$= \frac{s(t) + \epsilon\{\beta + \sigma\gamma\} + O(\epsilon^2)}{(4 - s(t)^2)\left(1 - \epsilon\frac{2s(t)\{\beta + \sigma\gamma\}}{4 - s(t)^2}\right) + O(\epsilon^2)}$$

which – after expanding the second factor in the denominator in series to first order in ϵ – becomes

$$\frac{s'(t')}{4 - s'(t')^2} = \frac{s(t)}{4 - s(t)^2} + \epsilon\{\beta + \sigma\gamma\}\left[\frac{4 + s(t)^2}{(4 - s(t)^2)^2}\right] + O(\epsilon^2).$$

Substituting these results into (9) yields

$$\mathcal{L}'\frac{dt'}{dt} = \hbar\left(\frac{d\chi(t)}{dt} + \epsilon\left\{\frac{d\alpha}{dt} + \frac{d(\eta\gamma)}{dt}\right\}\right) + i\hbar\left(\frac{s(t)}{4 - s(t)^2}\frac{ds(t)}{dt} + \epsilon\{\beta + \sigma\gamma\}\left[\frac{4 + s(t)^2}{(4 - s(t)^2)^2}\right]\frac{ds(t)}{dt} + \epsilon\left\{\frac{d\beta}{dt} + \frac{d(\sigma\gamma)}{dt}\right\}\frac{s(t)}{4 - s(t)^2}\right) + O(\epsilon^2)$$

or

$$\mathcal{L}'\frac{dt'}{dt} - \mathcal{L} = \epsilon\hbar\left[\frac{d\alpha}{dt} + \frac{d(\eta\gamma)}{dt} + i\left(\left\{\frac{d\beta}{dt} + \frac{d(\sigma\gamma)}{dt}\right\}\frac{s(t)}{4 - s(t)^2}\right) + i\{\beta + \sigma\gamma\}\left[\frac{4 + s(t)^2}{(4 - s(t)^2)^2}\right]\frac{ds(t)}{dt}\right] + O(\epsilon^2)\,.$$

Finally, comparing the last equation with (8) it is found that

$$\frac{df}{dt} = \hbar\left[\frac{d\alpha}{dt} + \frac{d(\eta\gamma)}{dt} + i\left(\left\{\frac{d\beta}{dt} + \frac{d(\sigma\gamma)}{dt}\right\}\frac{s(t)}{4 - s(t)^2}\right) + i\{\beta + \sigma\gamma\}\left(\frac{4 + s(t)^2}{(4 - s(t)^2)^2}\right)\frac{ds(t)}{dt}\right]. \quad (12)$$

Consequently, I is divergence invariant.

Here (12) can be rewritten as

$$\frac{df}{dt} = \hbar\frac{d}{dt}\left[(\alpha + \eta\gamma) + i(\beta + \sigma\gamma)\left(\frac{s(t)}{4 - s(t)^2}\right)\right]$$

so that

$$f = \hbar\left[(\alpha + \eta\gamma) + i(\beta + \sigma\gamma)\left(\frac{s(t)}{4 - s(t)^2}\right)\right] + C',$$

or– in terms of conjugate momenta –

$$f = p_\chi\alpha + \hbar\eta\gamma + (\beta + \sigma\gamma)p_s + C',$$

where C' is a complex valued integration constant.

A Conservation Law

Not only is I divergence invariant, it is also an extremal because $\mathcal{L}$ is a Lagrangian and necessarily satisfies the following Euler-Lagrange equations:

$$\frac{\partial\mathcal{L}}{\partial\chi(t)} = \frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial(d\chi(t)/dt)}\right)$$

and

$$\frac{\partial\mathcal{L}}{\partial s(t)} = \frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial(ds(t)/dt)}\right).$$

Because I is both divergence invariant and extremal, Noether's theorem can be used to provide a conservation law. This theorem states that if I is both extremal and divergence invariant, then (for this system) the conservation law

$$p_\chi\alpha + p_s\beta - f = C''$$

holds, where C'' is a complex valued constant. This becomes, after substitution for f and simplification,

$$\hbar\eta = Re\, C \equiv a \quad (13)$$

and

$$\sigma p_s = i\, Im\, C \equiv i\, b, \quad (14)$$

where $C = -(C'' + C') \equiv a + i\, b$. Here γ is set to unity and use is made of the facts that $\hbar\eta$ and σ are real valued and p_s is pure imaginary.

From (10) and (11) it is found that

$$\eta \equiv \left.\frac{d\chi'(t')}{dt'}\right|_0 = \frac{d\chi(t)}{dt}$$

and

$$\sigma \equiv \left.\frac{ds'(t')}{dt'}\right|_0 = \frac{ds(t)}{dt}$$

in which case (13) and (14) become

$$\hbar\frac{d\chi(t)}{dt} = a \qquad (15)$$

and

$$\hbar\frac{s(t)}{4-s(t)^2}\frac{ds(t)}{dt} = b\ . \qquad (16)$$

Thus, the conservation law implies the special case that $\mathcal{L}$ is a constant of the motion, i.e.

$$\mathcal{L} = a + i\,b.$$

Physical Consequences

Recall that, in general, $\mathcal{L}$ defines a multiplier that relates correlation amplitudes in time according to (Parks 2006)

$$\langle\psi(t)|\psi(0)\rangle = exp\left(\frac{i}{\hbar}\int_0^t \mathcal{L}\,dt'\right)\langle\psi(0)|\psi(0)\rangle$$

$$= exp\left(\frac{i}{\hbar}\int_0^t \mathcal{L}\,dt'\right)$$

since $\langle\psi(0)|\psi(0)\rangle = 1$. Because the conservation law requires $\mathcal{L}$ to be a constant of the motion, the last equation becomes

$$\langle\psi(t)|\psi(0)\rangle = exp\left\{\frac{i}{\hbar}\,(a + i\,b)\,t\right\}$$

(the integration constants are necessarily zero to satisfy the boundary condition $\langle\psi(0)|\psi(0)\rangle = 1$). Thus, the conservation law provides this simple expression for the correlation amplitude between an evolving state and the state at a previous initial time. It follows that the associated time dependent correlation probability is

$$|\langle\psi(t)|\psi(0)\rangle|^2 = exp\left\{-\frac{2b}{\hbar}t\right\}. \qquad (17)$$

When $\mathcal{L}$ is a constant of the motion the evolution of $|\psi\rangle$ relative to $|\psi_0\rangle$ in PFS configuration space can be determined by integrating (15) and (16) with respect to t to obtain

$$\chi(t) = \frac{at}{\hbar}$$

and

$$-\frac{\hbar}{2}\ln\left[1-\left(\frac{s(t)}{2}\right)^2\right] = bt$$

(the integration constants must vanish to satisfy the above boundary condition). The last equation can be solved for $s(t)$ to yield

$$s(t) = 2\sqrt{1-\exp\{-\frac{2b}{\hbar}t\}}\ .$$

This equation is in complete agreement with (2) when the exponential term in this equation is identified with the correlation probability in (17).

Closing Remarks

Noether's theorem and the associated ancillary theory developed in (Neuenschwander 2011) has been employed to formally derive the results found in this paper. Although these results were previously introduced by inspection without the benefit of Noether's theorem in (Parks 2003), this paper is a reaffirmation of the power and utility of Noether's theorem.

Appendix

The version of Noether's theorem used in this paper can be stated in terms of s, $\dot{\chi}$, and $\dot{s}$ as:

If the functional

$$I = \int \mathcal{L}(s(t),\dot{\chi}(t),\dot{s}(t))dt \equiv \int \mathcal{L}\,dt$$

is an extremal and if under the infinitesimal transformation

$$t' = t + \epsilon\gamma$$

$$\chi'(t) = \chi(t) + \epsilon\alpha$$

$$s'(t) = s(t) + \epsilon\beta$$

there exists a function $f \equiv f(t)$ such that

$$\mathcal{L}'\left(\frac{dt'}{dt}\right) - \mathcal{L} = \epsilon\frac{df}{dt} + O(\epsilon^r),\ \ r > 1,$$

where

$$\mathcal{L}' \equiv \hbar\frac{d\chi'(t')}{dt'} + i\hbar\frac{s'(t')}{4-s'(t')^2}\,\frac{ds'(t')}{dt'},$$

i.e. I is divergence invariant, then the conservation law

$$p_\chi\alpha + p_s\beta - [p_\chi\dot{\chi} + p_s\dot{s} - \mathcal{L}] - f = C''$$

or

$$p_\chi\alpha + p_s\beta - f = C''$$

holds. Here C'' is a complex valued constant and use is made of the fact that $\mathcal{L} = p_\chi\dot{\chi} + p_s\dot{s}$.

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