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Symmetry Equivalents of the Weak Value Measurement Pointer Hamiltonian

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Abstract: Quantum mechanical weak values and their measurement have been a focus of theoretical, experimental, and applied research for more than two decades. The concept of $\mathscr{PT}$ symmetry was also introduced into quantum mechanics during this time. This paper defines the notion of a weak value measurement pointer Hamiltonian and establishes equivalences between its Dirac symmetries, its $\mathscr{PT}$ symmetries, its eigenvalues, and the associated weak value. The affect of these symmetries upon measurement pointer observables is also identified.

Keywords: Weak values; pointer theory; symmetry; anti-symmetry.

Introduction

The weak value A_w of a quantum mechanical observable A was introduced a quarter century ago (Aharonov *et al* 1986; Aharonov *et al* 1988; Aharonov and Vaidman 1990). This quantity is the statistical result of a standard quantum measurement performed upon a pre- and postselected (PPS) ensemble of quantum systems when the interaction between the measurement apparatus and each system is sufficiently weak, that is, when it is a weak measurement. Unlike a standard strong measurement of A, which significantly disturbs the measured system (i.e., it "collapses" the wave function), a weak measurement of A for a PPS system does not appreciably disturb the quantum system and yields A_w as the observable's measured value. The peculiar nature of the virtually undisturbed quantum reality that exists between the boundaries defined by the PPS states is revealed by the eccentric characteristics of A_w, namely that A_w can be complex valued and that the real part Re A_w of A_w can lie far outside the eigenvalue spectral limits of $\hat{A}$ (this effect is called "weak value amplification").

While the interpretation of weak values remains somewhat controversial, experiments have verified several of the interesting unusual properties predicted by weak value theory (Ritchie *et al* 1991; Parks *et al* 1998; Resch *et al* 2004; Wang *et al* 2006; Hosten and Kwiat 2008; Yokota *et al* 2009; Dixon *et al* 2009; Parks and Spence 2016; Spence and Parks 2017).

Canonical quantum mechanics is based upon fundamental postulates, one of which requires that a quantum system's Hamiltonian operator $\hat{H}$ is Dirac symmetric, i.e. it is Hermitian, and obeys the symmetry condition $\hat{H} = \hat{H}^\dagger$ ("†" indicates adjoint). This means that the eigenvalues of $\hat{H}$ are real valued. $\hat{H}$ is anti-Dirac symmetric if $\hat{H}^\dagger = -\hat{H}$.

However, it has been shown (Bender and Boettcher 1999; Bender *et al* 1999) that certain non-Hermitian Hamiltonians, i.e., those for which $\hat{H}^\dagger \neq \pm\hat{H}$ and which generally have complex eigenvalues, will possess real valued eigenspectra if they exhibit the $\mathscr{PT}$ symmetry condition $\mathscr{PT}\hat{H} = \hat{H}$. Here $\mathscr{P}$ is the parity operator which changes the sign of both the position and momentum operators in $\hat{H}$ according to

$$\hat{q} \to -\hat{q}, \quad \hat{p} \to -\hat{p} \tag{1}$$

and $\mathcal{T}$ is the time reversal operator which leaves the position operator unchanged, changes the sign of the momentum operator, and performs complex conjugation in $\widehat{H}$ according to

$$\hat{q} \to \hat{q}, \qquad \hat{p} \to -\hat{p}, \qquad i \to -i \tag{2}$$

($i = \sqrt{-1}$). $\widehat{H}$ is anti- $\mathcal{PT}$ symmetric when $\mathcal{PT}\widehat{H} = -\widehat{H}$.

$\mathcal{PT}$ symmetric systems are known to occur in a variety of physical settings (Longhi 2010; Chtchelkatchev *et al* 2012; Hang *et al* 2013; Wu *et al* 2019) and have served as the basis for new devices (Benisty *et al* 2011) and novel metamaterials (Feng *et al* 2012).

A recently reported protocol for measuring $\mathcal{PT}$ symmetric non-Hermitian Hamiltonians using weak values (Pati *et al* 2015) has established an experimental connection between weak values and $\mathcal{PT}$ symmetric non-Hermitian Hamiltonians. This paper extends this connection by defining a special generally non-Hermitian weak value measurement pointer Hamiltonian $\widehat{H}_w$ and establishes equivalences between its Dirac symmetries, its $\mathcal{PT}$ symmetries, its eigenvalues, and its associated weak value. These symmetries are shown to manifest physically in the mean values and variances of measurement pointer observables.

Quantum Measurements and Weak Values

Standard Quantum Measurements

Before continuing, it is instructive to provide – in some detail - an overview of quantum measurement, weak measurement, and weak value theory. Weak measurements arise in the von Neumann description of an ideal quantum measurement at time t_0 of a time-independent observable A that describes a quantum system in an initial fixed preselected state $|\psi_i\rangle = \sum_J c_j\, |a_j\rangle$ at t_0, where the set J indexes the eigenstates $|a_j\rangle$ of the Hermitian operator $\hat{A}$ associated with the observable A (a_j is the eigenvalue satisfying the eigen equation $\hat{A}\,|a_j\rangle = a_j|a_j\rangle$). The Hamiltonian for the interaction between the measurement apparatus and the quantum system at t_0 is

$$\widehat{H} = \gamma(t)\hat{A}\hat{p}. \tag{3}$$

Here $\gamma(t) = \gamma\delta(t - t_0)$, where $\delta(t - t_0)$ is the Dirac delta function, defines the impulsive measurement interaction strength γ at t_0 and $\hat{p}$ is the momentum operator for the pointer of the measurement apparatus which is in the initial state $|\phi\rangle$. Let $\hat{q}$ be the pointer's position operator that is conjugate to $\hat{p}$.

Prior to the measurement, the preselected quantum system and the pointer are in the tensor product state $|\psi_i\rangle|\phi\rangle$. Immediately following the measurement, the combined system is in the state

$$|\Phi\rangle = e^{-\frac{i}{\hbar}\int \widehat{H}dt}|\psi_i\rangle|\phi\rangle = \sum\nolimits_J c_j\, e^{-\frac{i}{\hbar}\gamma a_j \hat{p}}|a_j\rangle|\phi\rangle,$$

where use has been made of the fact that $\int \widehat{H}dt = \gamma\hat{A}\hat{p}$ (recall that $\int \gamma\delta(t - t_0)dt = \gamma$). The exponential factor in this equation is the translation operator $\hat{S}(\gamma a_j)$ for $|\phi\rangle$ in its q representation. It is defined by the action

$$\langle q|\hat{S}(\gamma a_j)|\phi\rangle = \langle q - \gamma a_j|\phi\rangle \equiv \phi(q - \gamma a_j)$$

which translates the pointer's wavefunction over a distance γa_j parallel to the q axis. The q representation of the combined system and pointer state is

$$\langle q|\Phi\rangle = \sum\nolimits_J c_j\, \langle q|\hat{S}(\gamma a_j)|\phi\rangle|a_j\rangle.$$

For a standard quantum measurement, the measurement interaction is strong, the quantum system is appreciably disturbed and its state "collapses" to an eigenstate $|a_n\rangle$ leaving the pointer in the state $\langle q|\hat{S}(\gamma a_n)|\phi\rangle$ with probability $|c_n|^2$. Strong measurements of an ensemble of identically prepared systems yield $\gamma\langle\psi_i|\hat{A}|\psi_i\rangle$ as the centroid of the pointer probability distribution

$$|\langle q|\Phi\rangle|^2 = \sum\nolimits_J |c_j|^2\, |\langle q|\hat{S}(\gamma a_j)|\phi\rangle|^2 \tag{4}$$

with the mean value $\langle\psi_i|\hat{A}|\psi_i\rangle$ of $\hat{A}$ as the measured value of A.

Weak Measurements

A weak measurement of A occurs when the interaction strength γ is sufficiently small so that the system is essentially undisturbed and the pointer's position uncertainty Δq is much larger than $\hat{A}$'s eigenvalue separation. In this case, (4) is the superposition of broad overlapping $\left|\langle q|\hat{S}(\gamma a_j)|\phi\rangle\right|^2$ terms. Although a single measurement provides little information about A, many repetitions allow the centroid of (4) to be determined to any desired accuracy.

Postselected Weak Measurements

If a system state $|\psi_f\rangle = \sum_J c_j' \, |a_j\rangle$, $\langle\psi_f|\psi_i\rangle \neq 0$, at t_0 is postselected after a weak measurement is performed then the resulting pointer state is

$$|\Psi\rangle = \langle\psi_f|\Phi\rangle = \sum\nolimits_J c_j'^* \, c_j \hat{S}(\gamma a_j)|\phi\rangle$$

("*" denotes complex conjugate). Since

$$\hat{S}(\gamma a_j) \equiv e^{-\frac{i}{\hbar}\gamma a_j \hat{p}} = \sum_{m=0}^{\infty} \frac{\left[-\frac{i\gamma a_j \hat{p}}{\hbar}\right]^m}{m!},$$

then

$$|\Psi\rangle = \sum\nolimits_J c_j'^* \, c_j \left\{ \hat{1} - \frac{i}{\hbar}\gamma A_w \hat{p} + \sum_{m=2}^{\infty} \frac{\left[-\frac{i\gamma \hat{p}}{\hbar}\right]^m}{m!} (A^m)_w \right\} |\phi\rangle$$
$$\approx \left\{\sum\nolimits_J c_j'^* c_j\right\} e^{-\frac{i}{\hbar}\gamma A_w \hat{p}} |\phi\rangle \qquad (5)$$

in which case

$$|\Psi\rangle \approx \langle\psi_f|\psi_i\rangle \hat{S}(\gamma A_w)|\phi\rangle,$$

where

$$\hat{S}(\gamma A_w) = e^{-\frac{i}{\hbar}\gamma A_w \hat{p}},$$

so that

$$|\langle q|\Psi\rangle|^2 \approx \left|\langle\psi_f|\psi_i\rangle\right|^2 \left|\langle q|\hat{S}(\gamma A_w)|\phi\rangle\right|^2. \qquad (6)$$

Weak Values and the Weakness Conditions

In (5)

$$(A^m)_w = \frac{\sum_J c_j'^* c_j a_j^m}{\sum_J c_j'^* \, c_j} = \frac{\langle\psi_f|\hat{A}^m|\psi_i\rangle}{\langle\psi_f|\psi_i\rangle},$$

with the weak value A_w of A defined by

$$A_w \equiv (A^1)_w = \frac{\langle\psi_f|\hat{A}|\psi_i\rangle}{\langle\psi_f|\psi_i\rangle}.$$

It is obvious from this expression that A_w is in general a complex valued quantity that can be calculated directly from theory when the PPS states and $\hat{A}$ are known. Observe that if A_w and $\langle q|\phi\rangle$ are real valued, then (6) corresponds to a broad pointer position distribution with a single peak at $\langle\Psi|\hat{q}|\Psi\rangle = \gamma Re\, A_w$ with $Re\, A_w$ as the measured weak value of A. This occurs when both of the following inequalities, i.e. the weakness conditions, which relate γ, A_w, and the pointer momentum uncertainty Δp, are satisfied (Parks and Spence 2017)

$$\Delta p \ll \frac{\hbar}{\gamma}|A_w|^{-1} \text{ and } \Delta p \ll \min_{(m=2,3,\cdots)} \frac{\hbar}{\gamma}\left|\frac{A_w}{(A^m)_w}\right|^{(m-1)^{-1}}.$$

The Weak Value Measurement Pointer Hamiltonian

Observe that a new Hamiltonian can be formed from (3) by simply replacing $\hat{A}$ with A_w to obtain *the weak value measurement pointer Hamiltonian*

$$\widehat{H}_w = \gamma(t) A_w \hat{p}.$$

This Hamiltonian is strictly associated with the dynamics of the pointer (as indicated by $\hat{p}$) at measurement time t_0 (because it is constrained by $\delta(t-t_0)$) after postselection (as required by A_w). It is the symmetries of this Hamiltonian that are found below. Note that because $\gamma(t)$ is real valued and A_w is generally complex valued, then $\widehat{H}_w$ is generally a non-Hermitian operator because

$$\widehat{H}_w^\dagger = (\gamma(t)A_w\hat{p})^\dagger = \gamma(t)A_w^*\hat{p}^\dagger = \gamma(t)A_w^*\hat{p} \neq \widehat{H}_w.$$

It is interesting to determine the eigenspectrum and eigenfunctions of $\widehat{H}_w$. Since $\widehat{H}_w$ is effectively the momentum operator scaled by $\gamma(t)A_w$, it is expected that its eigenstates are momentum eigenstates and its eigenvalues are scaled momentum values.

To see that this is the case, consider the associated eigen equation

$$\widehat{H}_w|\varphi\rangle = \lambda|\varphi\rangle.$$

In the q-representation this becomes

$$\langle q|\widehat{H}_w|\varphi\rangle = \lambda\langle q|\varphi\rangle$$

or

$$\gamma(t)A_w\langle q|\hat{p}|\varphi\rangle = \lambda\langle q|\varphi\rangle.$$

Applying the identities $\langle q|\varphi\rangle = \varphi(q)$ and $\langle q|\hat{p}|\varphi\rangle = \frac{\hbar}{i}\frac{d\varphi(q)}{dq}$ to this yields the differential equation

$$\frac{\hbar}{i}\gamma(t)A_w\,\frac{d\varphi(q)}{dq} = \lambda\varphi(q).$$

The solution to this equation is the momentum eigenstate

$$\varphi(q) = \frac{1}{\sqrt{2\pi\hbar}}e^{\frac{i}{\hbar}qp}$$

which - by inspection - yields the eigenvalue

$$\lambda = \gamma(t)A_w p.$$

Because $\gamma(t)$ and p are real valued and A_w is generally complex valued, λ is also a generally complex valued quantity. As above, since λ depends upon $\delta(t-t_0)$, it is the eigenvalue at measurement time t_0.

Symmetry and Anti-Symmetry Equivalents Theorems

Symmetry equivalents of the weak value measurement pointer Hamiltonian are identified in the following theorem:

Theorem 1. *The following statements are equivalent:*

(1) A_w *is real valued;*
(2) $\widehat{H}_w$ *is Dirac symmetric (Hermitian);*
(3) $\widehat{H}_w$ *is* $\mathscr{PT}$ *symmetric;*
(4) λ *is real valued.*

Proof.
(1) ⇔ (2): A_w real ⇔ $A_w^* = A_w$ ⇔ $\gamma(t)A_w^*\hat{p} = \gamma(t)A_w\hat{p}$ ⇔ $\widehat{H}_w^\dagger = \widehat{H}_w$.
(2) ⇔ (3): $\widehat{H}_w = \widehat{H}_w^\dagger$ ⇔ $\gamma(t)A_w\hat{p} = \gamma(t)A_w^*\big(-(-\hat{p})\big)$ ⇔ $\widehat{H}_w = \mathscr{PT}\,\widehat{H}_w$.
(3) ⇔ (4): $\widehat{H}_w = \mathscr{PT}\,\widehat{H}_w$ ⇔ $\gamma(t)A_w\hat{p} = \gamma(t)A_w^*\big(-(-\hat{p})\big)$ ⇔ $\gamma(t)A_w\hat{p} = \gamma(t)A_w^*\hat{p}$ ⇔ $\gamma(t)A_w\hat{p}|\varphi\rangle = \gamma(t)A_w^*\hat{p}|\varphi\rangle$ ⇔ $\gamma(t)A_w\langle q|\hat{p}|\varphi\rangle = \gamma(t)A_w^*\langle q|\hat{p}|\varphi\rangle$ ⇔ $\gamma(t)A_w\,\frac{\hbar}{i}\frac{d\varphi(q)}{dq} = \gamma(t)A_w^*\,\frac{\hbar}{i}\frac{d\varphi(q)}{dq}$ ⇔ $\gamma(t)A_w p\varphi(q) = \gamma(t)A_w^* p\varphi(q)$ ⇔ $\gamma(t)A_w p = \gamma(t)A_w^* p$ ⇔ $\lambda = \lambda^*$ ⇔ λ is real valued.
(4) ⇔ (1): λ real ⇔ $\lambda = \lambda^*$ ⇔ $\gamma(t)A_w p = \gamma(t)A_w^* p$ ⇔ $A_w = A_w^*$ ⇔ A_w is real. ■

When any item in Theorem 1 holds, then $\widehat{H}_w$ is said to be symmetric.

The anti-symmetric equivalents of the weak value measurement pointer Hamiltonian are delineated in the next theorem:

Theorem 2. *The following statements are equivalent:*

(1) A_w *is pure imaginary;*
(2) $\widehat{H}_w$ *is anti-Dirac symmetric (anti-Hermitian);*
(3) $\widehat{H}_w$ *is anti-*$\mathscr{PT}$ *symmetric;*
(4) λ *is pure imaginary.*

Proof.
(1) ⇔ (2): A_w pure imaginary ⇔ $A_w^* = -A_w$ ⇔ $\gamma(t)A_w^*\hat{p} = -\gamma(t)A_w\hat{p}$ ⇔ $\widehat{H}_w^\dagger = -\widehat{H}_w$.
(2) ⇔ (3): $-\widehat{H}_w = \widehat{H}_w^\dagger$ ⇔ $-\gamma(t)A_w\hat{p} = \gamma(t)A_w^*\big(-(-\hat{p})\big)$ ⇔ $-\widehat{H}_w = \mathscr{PT}\,\widehat{H}_w$.
(3) ⇔ (4): $\widehat{H}_w = -\mathscr{PT}\,\widehat{H}_w$ ⇔ $\gamma(t)A_w\hat{p} = -\gamma(t)A_w^*\big(-(-\hat{p})\big)$ ⇔ $\gamma(t)A_w\hat{p} = -\gamma(t)A_w^*\hat{p}$ ⇔ $\gamma(t)A_w\hat{p}|\varphi\rangle = -\gamma(t)A_w^*\hat{p}|\varphi\rangle$ ⇔ $\gamma(t)A_w\langle q|\hat{p}|\varphi\rangle = -\gamma(t)A_w^*\langle q|\hat{p}|\varphi\rangle$ ⇔ $\gamma(t)A_w\,\frac{\hbar}{i}\frac{d\varphi(q)}{dq} = -\gamma(t)A_w^*\,\frac{\hbar}{i}\frac{d\varphi(q)}{dq}$ ⇔ $\gamma(t)A_w p\varphi(q) = -\gamma(t)A_w^* p\varphi(q)$ ⇔ $\gamma(t)A_w p = -\gamma(t)A_w^* p$ ⇔ $\lambda = -\lambda^*$ ⇔ λ is pure imaginary.
(4) ⇔ (1): λ pure imaginary ⇔ $\lambda = -\lambda^*$ ⇔

$\gamma(t)A_w p = -\gamma(t)A_w^* p \Leftrightarrow A_w = -A_w^* \Leftrightarrow$
A_w is pure imginary. ■

When any item in Theorem 2 holds, then $\widehat{H}_w$ is said to be anti-symmetric.

Note that in the above proofs "⇔" means "if, and only if" and use is made of the fact that $\hat{p}^\dagger = \hat{p}$.

Affect of $\widehat{H}_w$ Symmetries and Anti-Symmetries Upon Pointer Observables

The general results previously reported in the literature (Josza 2007; Parks and Gray 2014) are used here to show how the symmetries and anti-symmetries of $\widehat{H}_w$ affect measured pointer observables after a weak value measurement of an observable A. Let $\hat{O}$ be the operator associated with a pointer observable O. After a weak value measurement of A, the mean value and variance of $\hat{O}$ are - in general - given by

$$\langle\Psi|\hat{O}|\Psi\rangle = \langle\phi|\hat{O}|\phi\rangle - i\left(\frac{\gamma}{\hbar}\right) Re\, A_w \langle\phi|[\hat{O},\hat{p}]|\phi\rangle + \left(\frac{\gamma}{\hbar}\right) Im\, A_w\left(\langle\phi|\{\hat{O},\hat{p}\}|\phi\rangle - 2\langle\phi|\hat{O}|\phi\rangle\langle\phi|\hat{p}|\phi\rangle\right) \quad (7)$$

and

$$\Delta_\Psi^2 O = \Delta_\phi^2 O - i\left(\frac{\gamma}{\hbar}\right) Re\, A_w\, F(\hat{O}) + \left(\frac{\gamma}{\hbar}\right) Im\, A_w\, G(\hat{O}), \quad (8)$$

respectively. Here, $\Delta_Y^2 O$ is the variance of $\hat{O}$ for pointer state $Y = \Psi, \phi$;

$$F(\hat{O}) = \langle\phi|[\hat{O}^2,\hat{p}]|\phi\rangle - 2\langle\phi|\hat{O}|\phi\rangle\langle\phi|[\hat{O},\hat{p}]|\phi\rangle;$$

$$G(\hat{O}) = \langle\phi|\{\hat{O}^2,\hat{p}\}|\phi\rangle - 2\langle\phi|\hat{O}|\phi\rangle\langle\phi|\{\hat{O},\hat{p}\}|\phi\rangle - 2\langle\phi|\hat{p}|\phi\rangle\left(\Delta_\phi^2 O - \langle\phi|\hat{O}|\phi\rangle^2\right);$$

$$[\hat{X},\hat{p}] = (\hat{X}\hat{p} - \hat{p}\hat{X});$$

and

$$\{\hat{X},\hat{p}\} = \hat{X}\hat{p} + \hat{p}\hat{X},$$

where $\hat{X} = \hat{O}, \hat{O}^2$. Recall that $|\Psi\rangle$ is the pointer state after the measurement and $|\phi\rangle$ is the pointer state before the measurement.

Observe that when $\widehat{H}_w$ is symmetric (Theorem 1), then $Im\, A_w = 0$ and (7) and (8) become

$$\langle\Psi|\hat{O}|\Psi\rangle = \langle\phi|\hat{O}|\phi\rangle - i\left(\frac{\gamma}{\hbar}\right) Re\, A_w \langle\phi|[\hat{O},\hat{p}]|\phi\rangle \quad (9)$$

and

$$\Delta_\Psi^2 O = \Delta_\phi^2 O - i\left(\frac{\gamma}{\hbar}\right) Re\, A_w\, F(\hat{O}), \quad (10)$$

respectively (obviously, in this case $Re\, A_w = A_w$), whereas when $\widehat{H}_w$ is anti-symmetric (Theorem 2), then Re $A_w = 0$ and (7) and (8) reduce to

$$\langle\Psi|\hat{O}|\Psi\rangle = \langle\phi|\hat{O}|\phi\rangle + \left(\frac{\gamma}{\hbar}\right) Im\, A_w\left(\langle\phi|\{\hat{O},\hat{p}\}|\phi\rangle - 2\langle\phi|\hat{O}|\phi\rangle\langle\phi|\hat{p}|\phi\rangle\right) \quad (11)$$

and

$$\Delta_\Psi^2 O = \Delta_\phi^2 O + \left(\frac{\gamma}{\hbar}\right) Im\, A_w\, G(\hat{O}), \quad (12)$$

respectively (obviously, in this case $Im\, A_w = A_w$). Comparison of the differences between (9) and (11) and between (10) and (12) reveals the obvious dependence of the mean values and variances of $\hat{O}$ upon the symmetry and anti-symmetry of $\widehat{H}_w$.

Before leaving this section, it is worth noting that if $\widehat{H}_w$ is symmetric and $\hat{O} = \hat{q}$ (a pointer observable of frequent interest), then (9) becomes

$$\langle\Psi|\hat{q}|\Psi\rangle = \langle\phi|\hat{q}|\phi\rangle + \gamma Re\, A_w \quad (13)$$

and, because

$$F(\hat{q}) = \langle\phi|[\hat{q}^2,\hat{p}]|\phi\rangle - 2\langle\phi|\hat{q}|\phi\rangle\langle\phi|[\hat{q},\hat{p}]|\phi\rangle = 0,$$

(10) becomes

$$\Delta_\Psi^2 q = \Delta_\phi^2 q. \quad (14)$$

Here, use is made of the facts that $[\hat{q},\hat{p}] = i\hbar$ and $[\hat{q}^2,\hat{p}] = 2i\hbar\hat{q}$. From (13) and (14) it is seen that the measurement shifts the pointer from its initial premeasurement mean position by $\gamma Re\, A_w$ without changing the associated variance.

Recall that (13) corresponds to the weak value amplification effect when $Re\, A_w$ exceeds the eigen value spectral limit of $\hat{A}$. This effect can be used to shift the pointer position sufficiently far from $\langle\phi|\hat{q}|\phi\rangle$ so that very small coupling coefficients can be measured according to

$$\gamma = \frac{\langle\Psi|\hat{q}|\Psi\rangle - \langle\phi|\hat{q}|\phi\rangle}{Re\ A_w}$$

when $Re\ A_w$ can be calculated directly from theory. Condition (14) is especially important for this measurement when the initial variance can be "tuned" to be small.

Closing Remarks

The von Neumann measurement interaction Hamiltonian operator was used to define a new Hamiltonian – the weak value measurement pointer Hamiltonian – $\widehat{H}_w$. This Hamiltonian is strictly associated with the dynamics of the measurement pointer.

Two theorems were developed which establish symmetric and anti-symmetric equivalents for $\widehat{H}_w$. It was shown that symmetric and anti-symmetric equivalents affect the mean values and variances of pointer observables in distinct and useful ways. For example, symmetric $\widehat{H}_w$'s can serve as weak value amplifiers of pointer position without increasing the initial pointer position variance. The utility of this method is an enhanced measurement accuracy of the values of small interaction strengths of physical interest which are unobtainable by other methods.

As a final remark, the following relationship between the pointer translation operator and $\widehat{H}_w$ is noted:

$$\hat{S}(\gamma A_w) = e^{-\frac{i}{\hbar}\int \widehat{H}_w dt}.$$

This expression justifies the above claim that $\widehat{H}_w$ is strictly associated with the dynamics of the pointer.

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